STUDY MATERIALS

Suri Vidyasagar College
Department of Mathematics

SEMESTER – I (Major)

Course Type – CC

Course Code – BMH1CC02

Course Name: Analytical Geometry (2D)

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General Equation of Second degree

Introduction :- The general equation of second degree in x and y is usually written in the form —

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad ---(1)$$

The curve represented by this equation is a conic section or simply a conic. The curve is also called a second order curve. The nature of the conic is determined by the quantities

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad D = ab-h^2 \text{ and } P = a+b$$

In case of rectangular co-ordinate axes the quantities Δ, D and P are invariants under any orthogonal transformation.

i) If $\Delta = 0$, the equation (1) represents a pair of straight lines.

ii) If $a=b$ and $h=0$, the equation represents a circle.

iii) If $\Delta \neq 0$, the equation represents a proper conic.
Here D determines the nature of the conic.

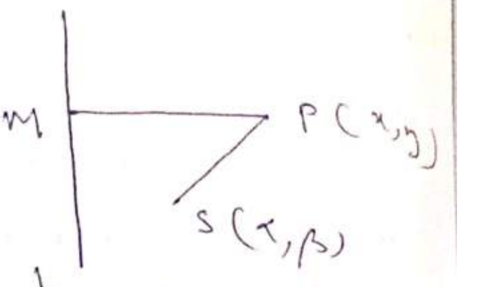
(a) - When $D=0$, i.e. $ab=h^2$, the conic is a parabola. In this case the second degree terms form a perfect square.

(b) When $D>0$, i.e. $ab>h^2$, the conic is an ellipse.

(c) - When $D<0$, i.e $ab<h^2$, the conic is a hyperbola. If $a+b=0$, the conic is a rectangular hyperbola.

Conditions for proper conics :-

Let LM be the directrix, $S(\alpha,\beta)$ be the focus and $P(x,y)$ be any point on the conic whose eccentricity is e.



The equation of LM is $lx+my+n=0$. If PM is perpendicular to the directrix then by definition of a conic,

$$SP^2 = e^2 PM^2,$$

or $$(x-\alpha)^2+(y-\beta)^2 = e^2\frac{(lx+my+n)^2}{l^2+m^2},$$

or $$\{l^2(1-e^2)+m^2\}x^2 - 2lme^2xy + \{l^2+m^2(1-e^2)\}y^2 - 2\{(l^2+m^2)\alpha+lne^2\}x - 2\{(l^2+m^2)\beta+mne^2\}y$$
$$(l^2+m^2)(\alpha^2+\beta^2) - n^2e^2 = 0.$$

If this equation represents the equation (1) then we can write that

$$a = l^2(1-e^2)+m^2,\quad b = l^2+m^2(1-e^2),\quad h = -lme^2,$$
$$g = -\{(l^2+m^2)\alpha + lne^2\},\quad f = -\{(l^2+m^2)\beta+mne^2\}$$
$$c = (l^2+m^2)(\alpha^2+\beta^2) - n^2e^2.$$

Now $$D = ab-h^2 = \{l^2(1-e^2)+m^2\}\{l^2+m^2(1-e^2)\} - l^2m^2e^4$$
$$= (l^2+m^2)^2(1-e^2)$$

For the parabola, $e=1$, i.e. $D=0$
For the ellipse, $e<1$, i.e. $D>0$
For the hyperbola, $e>1$, i.e. $D<0$

Canonical form :-

The general equation of a second degree can be reduced to the standard equation of a conic by suitable transformation of co-ordinates. This standard equation is also called the canonical equation or normal canonical form of the equation.

To find the canonical form from the general equation the following transformations are made successively.

(i) The term in xy is removed by suitable rotation of axes.

(ii) One or both (when possible) the terms in x and y are removed by translation.

(iii) The constant is removed if possible.

Reduction to canonical form (Tracing of conic):

Let the axes be rotated through an angle θ. The new co-ordinates are related as

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Substituting these values of x and y in the eqn (1) we have —

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta - y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta) + c = 0$$

or
$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2 + 2\{h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta\}x'y' + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 + 2(g\cos\theta + f\sin\theta)x' + 2(f\cos\theta - g\sin\theta)y' + c = 0 \quad (2)$$

Let us choose θ in such a way that the co-efficient of $x'y'$ in (2) will be zero. To satisfy this condition, we have

$$h(\cos^2\theta - \sin^2\theta) = (a-b)\sin\theta\cos\theta,$$

or, $\tan 2\theta = \dfrac{2h}{a-b}$ i.e. $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2h}{a-b}$

For this value of θ the equation (2) will be of the form

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + C = 0 \quad \text{——} (3)$$

By property of invariants,

$$\Delta = \begin{vmatrix} A & 0 & G \\ 0 & B & F \\ G & F & C \end{vmatrix}, \quad D = AB \text{ and } P = A+B$$

⟨I⟩ If $\Delta \neq 0$, but $D = 0$ then the eqn (1) represents a parabola.

There are three possibilities

(i) $A = 0$, $B = 0$

(ii) $A = 0$, $B \neq 0$

(iii) $A \neq 0$, $B = 0$

If $A = 0$ and $B = 0$, then $\Delta = 0$. Possibility (i) is thus ruled out. For the possibility (ii) $\Delta \neq 0$, if $G \neq 0$

In this case the equation (3) reduces to

$$By'^2 + 2Gx' + 2Fy' + C = 0 \quad \text{——} (4)$$

It is a parabola having its axis parallel to the new x-axis.

From (4), $B\left(y'^2 + \frac{F}{B}y'\right) = -2Gx' - C$

or $$\left(y' + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x' + \frac{BC - F^2}{2GB}\right).$$

Changing the origin to $\left(-\frac{BC - F^2}{2GB}, -\frac{F}{B}\right)$, the equation reduces to

$$y''^2 = -\frac{2G}{B}x'' \quad \text{——} (5)$$

It is the canonical form of the equation (1) when $\Delta \neq 0$ but $D = 0$

For the possibility (iii) $A \neq 0$, if $F \neq 0$.

In this case, the equation (3) reduces to

$$Ax'^2 + 2Gx' + 2Fy' + C = 0. \quad \text{——— (6)}$$

It is a parabola having its axis parallel to the y'-axis i.e. new y-axis.

From (6), $x'^2 + \frac{2G}{A}x' = -\frac{2F}{A}y' - \frac{C}{A}$,

$$\text{or } \left(x' + \frac{G}{A}\right)^2 = -\frac{2F}{A}\left(y' + \frac{CA - G^2}{2FA}\right)$$

Changing the origin to $\left(-\frac{G}{A}, -\frac{CA-G^2}{2FA}\right)$, the equation reduces to

$$x''^2 = -\frac{2F}{A}y''. \quad \text{——— (7)}$$

It is the canonical form of the equation (1).

Note-1. Since $h^2 = ab$, the terms of second degree in the equation (1) form a perfect square.

Note-2. If $A = 0 = B$, the $a = 0 = b = h$ and the equation (1) represents a line.

Note-3. If $A = 0 = G$, but $B \neq 0$, then the eqn (3)

$$By'^2 + 2Fy' = 0 \text{ or } \left(y' + \frac{F}{B}\right)^2 = \frac{F^2 - BC}{B^2}$$

Here the equation (1) represents a pair of parallel lines, a pair of coincident lines or no geometric locus according as $F^2 - BC >=< 0$.

Note-4. If $B = 0 = F$ but $A \neq 0$, then the eqn (3) reduces to $Ax'^2 + 2Gx' + C = 0$, or $\left(x' + \frac{G}{A}\right)^2 = \frac{G^2 - CA}{A^2}$

Here the equation (1) represents a pair of parallel lines, a pair of coincident lines or no geometric locus according as $G^2 - CA >=< 0$.

Example :- Discuss the nature of the conic represented by $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$ and reduces it to the canonical form (normal form).

Solⁿ :- Here $a = 9,\ h = -12,\ b = 16,\ g = -9,\ f = -\frac{101}{2}$
$c = 19$,

$$\Delta = \begin{vmatrix} 9 & -12 & -9 \\ +12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 19 \end{vmatrix} \neq 0, \quad D = 9.16 - (12)^2 = 0$$

Therefore the given equation represents a parabola.
Let the axes be rotated through an acute angle θ. The equation transforms to

$$9(x'\cos\theta - y'\sin\theta)^2 - 18(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$+16(x'\sin\theta + y'\cos\theta)^2 - 18(x'\cos\theta - y'\sin\theta) - 101(x'\sin\theta$$
$$+ y'\cos\theta) + 19 = 0$$

or, $$9\cos^2\theta - 24\sin\theta\cos\theta + 16\sin^2\theta)x'^2 - 2\{12(\cos^2\theta - \sin^2\theta)$$
$$-7\sin\theta\cos\theta\}x'y' + (16\cos^2\theta + 24\sin\theta\cos\theta + 9\sin^2\theta)y'^2$$
$$-(18\cos\theta + 101\sin\theta)x' + (18\sin\theta - 101\cos\theta)y' + 19 = 0$$

or $$(3\cos\theta - 4\sin\theta)^2 x'^2 - 2\{12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta\}x'y'$$
$$+(4\cos\theta + 3\sin\theta)^2 y'^2 - (18\cos\theta + 101\sin\theta)x' + (18\sin\theta - 101\cos\theta)y'$$
$$+19 = 0$$

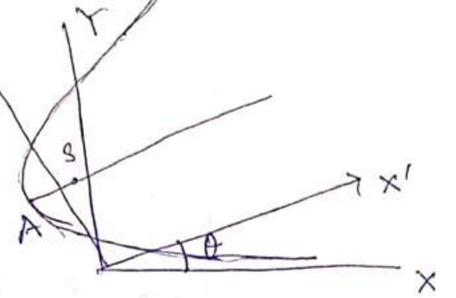
Let us choose θ in such a way that
$$12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta = 0$$

From this $12\tan^2\theta + 7\tan\theta - 12 = 0$ or $\tan\theta = \frac{3}{4}, -\frac{4}{3}$
Since θ is acute, $\tan\theta = \frac{3}{4}$. Hence $\sin\theta = \frac{3}{5}$ and $\cos\theta = 4/5$.

For these values of $\cos\theta$ and $\sin\theta$ the equation reduces to
$$25y'^2 - 75x' - 70y' + 19 = 0$$

or $(y' - 7/5)^2 = 3(x' + 2/5)$

Changing the origin to $(-2/5, 7/5)$, the equation further reduces to $y''^2 = 3x''$. It is the required canonical form.



The parabola can be traced now as shown in the figure. In the figure, $\theta = \tan^{-1} 3/4$; Vertex A is at $(-2/5, 7/5)$ w.r.t. OX', OY' and $AS = 3/4$, S being the focus of the parabola.

⟨II⟩ The equation (1) represents an ellipse when $\Delta \neq 0$ but $D > 0$

We have $D = AB$, If $D > 0$, none of A and B is zero and both of them are positive or negative. Without any loss of generality we may assume that both of A and B are positive. From (3)

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = K \text{ (say)}.$$

By translation $x' = x'' - \frac{G}{A}$, $y' = y'' - \frac{F}{B}$, the equation reduces to

$$Ax''^2 + By''^2 = K$$

$$\text{or, } \frac{x''^2}{K/A} + \frac{y''^2}{K/B} = 1 \quad \text{——} (8)$$

It is the equation of the ellipse in the canonical form.

Note-(1):- The centre of the ellipse represented by the equation (1) is

$$\left(-\frac{G}{A}\cos\theta + \frac{F}{B}\sin\theta,\ -\frac{G}{A}\sin\theta - \frac{F}{B}\cos\theta\right)$$

Note-2:- The eqn (8) represents a real ellipse, a point (null) ellipse or an imaginary (without any real trace) ellipse according as $K >=< 0$.

Note-3:- $\Delta = -ABK$. It is $<=>0$ according as the eqn (8) represents a real ellipse, a point ellipse or an imaginary ellipse.

Example:- Reduce the equation $3x^2+2xy+3y^2-16x+20=0$ to normal form.

Soln:- Here $a=3, h=1, b=3, g=-8, f=0, c=20$

$$\Delta = \begin{vmatrix} 3 & 1 & -8 \\ 1 & 3 & 0 \\ -8 & 0 & 20 \end{vmatrix} = -32 \neq 0 \text{ and } D = 3.3-1^2 = 8 > 0.$$

Therefore the given equation represents an ellipse.

By rotating the axes through an acute angle θ, the equation transforms to -

$$3(x'\cos\theta - y'\sin\theta)^2 + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 3(x'\sin\theta + y'\cos\theta)^2 - 16(x'\sin\theta - y'\cos\theta) + 20 = 0$$

or $(3+\sin 2\theta)x'^2 + 2\cos 2\theta\, x'y' + (3-\sin 2\theta)y'^2 - 16\sin\theta\, x' + 16\sin\theta\, x' + 16\cos\theta\, y' + 20 = 0$

θ is chosen in such a way that $\cos 2\theta = 0$, $2\theta = 90°$ or $\theta = 45°$.

For this value of θ, the equation takes the form

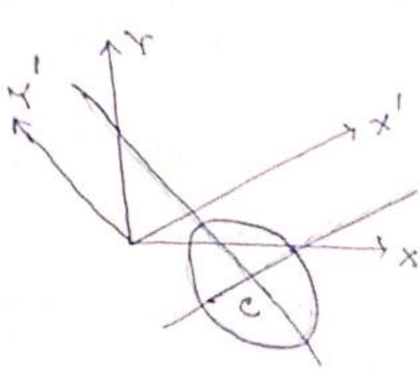
$$4x'^2 + 2y'^2 - 8\sqrt{2}x' + 8\sqrt{2}y' + 20 = 0$$

or $4(x'-\sqrt{2})^2 + 2(y'+2\sqrt{2})^2 = 4$

Changing the origin to $(\sqrt{2}, -2\sqrt{2})$, the equation reduces to

$$4x''^2 + 2y''^2 = 4$$

or $x''^2 + y''^2/2 = 1$



It is the required normal form. The conic is an ellipse with semi-axes 1 and $\sqrt{2}$.

The ellipse can be traced now as shown in figure. In the figure $\theta = 45$, the centre c is at $(\sqrt{2}, -2\sqrt{2})$ w.r.t. Ox', Oy'.

(III) The equation (1) represents a hyperbola when $\Delta \neq 0$. but $D < 0$:-

If $D < 0$ then $AB < 0$. Consequently none of A and B is zero. Moreover A and B have opposite signs. Without any loss of generality we may assume that $A > 0$ and $B < 0$. Proceeding as (II) The equation (3) reduces to -

$$\frac{x''^2}{K/A} + \frac{y''^2}{K/B} = 1 \quad \text{when } K \text{ is not zero.}$$

If $K > 0$ then it can be written as

$$\frac{x''^2}{\alpha^2} - \frac{y''^2}{\beta^2} = 1 \quad \cdots (9)$$

It is the equation of the hyperbola in the canonical form

If $K < 0$ then the equation can be written as

$$\frac{x''^2}{\alpha^2} - \frac{y''^2}{\beta^2} = -1 \quad \text{——} (10)$$

It is also the canonical form.

Note :- (1) Hyperbola represented by (9) and (10) are conjugate to each other.

Note :- (2) The centre of the hyperbola represented by the equation (1) is

$$\left(-\frac{G}{A}\cos\theta + \frac{F}{B}\sin\theta, -\frac{G}{A}\sin\theta - \frac{F}{B}\cos\theta\right)$$

Note-(3) :- If $K = 0$, The equation (3) reduces to $Ax''^2 + By''^2 = 0$. It represents a pair of straight lines which are asymptotes to the hyperbolas represented by (9) and (10)

Note-4 :- If $\alpha^2 = \beta^2$, The hyperbola is rectangular.

Note-5 :- Here $\Delta = -ABK$. It may be $<$ or > 0 for the real hyperbola.

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Example :- Show that the equation

$$7x^2 - 48xy - 7y^2 - 20x + 140y + 300 = 0$$

represents a hyperbola and find its canonical form

Solutions— Here $a = 7$, $h = -24$, $b = -7$, $g = -10$, $f = 70$

$c = 300$,

$$\Delta = \begin{vmatrix} 7 & -24 & -10 \\ -24 & -7 & 70 \\ -10 & 70 & 300 \end{vmatrix} \neq 0, \quad D = 7(-7) - (-24)^2 < 0.$$

Therefore the given equation represents a hyperbola.

Rotating the axes through an acute angle θ, the equation transforms to

$$7(x'\cos\theta - y'\sin\theta)^2 - 48(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$- 7(x'\sin\theta + y'\cos\theta)^2 - 20(x'\cos\theta - y'\sin\theta) + 140(x'\sin\theta +$$
$$y'\cos\theta) + 300 = 0$$

or $(7\cos^2\theta - 7\sin^2\theta - 48\sin\theta\cos\theta)x'^2 - \{48(\cos^2\theta - \sin^2\theta)$
$+ 28\sin\theta\cos\theta\}x'y' - (7\cos^2\theta - 7\sin^2\theta - 48\sin\theta\cos\theta)y'^2$
$- 20(\cos\theta - 7\sin\theta)x' + 20(\sin\theta + 7\cos\theta)y' + 300 = 0$

To make the coefficient of $x'y'$ zero,

$$48(\cos^2\theta - \sin^2\theta) + 28\sin\theta\cos\theta = 0,$$

or $12\tan^2\theta - 7\tan\theta - 12 = 0$

or $\tan\theta = \frac{4}{3}, -3/4.$

Taking $\tan\theta = \frac{4}{3}$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

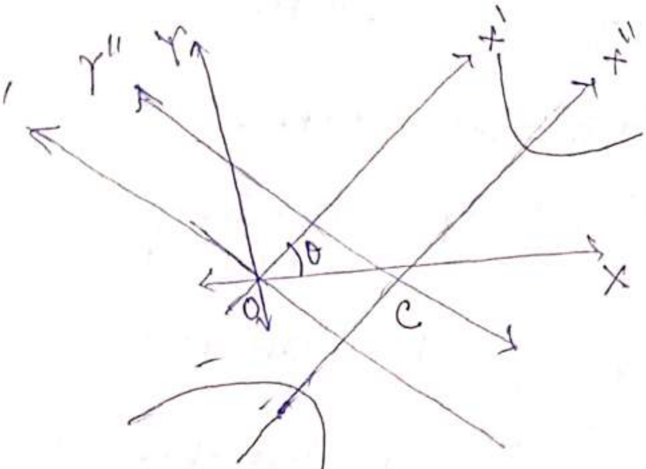
For these values of $\sin\theta$ and $\cos\theta$, the above equation takes the form

$$y'^2 - x'^2 + 4x' + 4y' + 12 = 0$$

or $(x'-2)^2-(y'+2)^2=12$

By translation, $x'=x''+2$, $y'=y''-2$, the equation reduces to $x''^2-y''^2=12$

It is the canonical equation of the hyperbola which is rectangular one in this case. The rectangular hyperbola can be traced now as shown in figure. In the figure $\theta=\tan^{-1}4/3$, the centre C is at $(2,-2)$ w.r.t. OX', OY'.



Rank and classification of a second ~~degree~~ order curve:-

The rank of a second order curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ is the rank of the matrix $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$.

A second order curve is classified as non-singular or non-degenerate and singular or degenerate according as its rank is 3 and and 2 or 1 respectively. A circle a parabola, an ellipse and a hyperbola are non-singular. A point ellipse, a pair of intersecting lines, a pair of parallel lines and a pair of coincident lines are singular or degenerate. The rank of a pair of coincident lines is one.

Note:- The general second degree equation represents (i) a degenerate conic if $\Delta=0$ and (ii) a non-degenerate conic if $\Delta\neq 0$.

The non-degenerate conics are mainly divided into three classes:

(a) elliptic if $D>0$, (b) parabolic if $D=0$ and (c) hyperbolic if $D<0$.

Circle is a special case of the ellipse when the major and minor axes are equal.

Table for metric classification :-

Δ	D	canonical form	Name	Rank	class
$\Delta<0$	$D>0$	$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta}=1$	ellipse	3	non-singular
$\Delta<0$	$D<0$	$x^2+y^2=\alpha^2$	circle	3	non-singular
$\Delta>0$	$D>0$	$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$	Imaginary ellipse	3	non-singular
$\Delta>0$	$D<0$	$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$	hyperbola	3	non-singular
$\Delta<0$	$D<0$	$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=-1$	hyperbola	3	non-singular
$\Delta\neq0$	$D=0$	$y^2=4\alpha x$ $x^2=4\beta y$	parabola	3	non-singular
$\Delta=0$	$D>0$	$Ax^2+By^2=0$	Pair of imaginary lines or point ellipse	2	Singular
$\Delta=0$	$D<0$	$y^2-k^2x^2=0$	pair of intersecting lines	2	Singular
$\Delta=0$	$D=0$	$y^2=k^2$ $x^2=l^2$	pair of parallel lines	2	Singular
$\Delta=0$	$D=0$	$y^2=0$ $x^2=0$	Pair of Coincident lines	1	Singular